A switching model of dynamic asset selling problem : the case of multiple homogeneous assets

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A switching model of dynamic asset selling problem
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Abstract This paper proposes a new selling strategy called the *switching strategy* where a seller who wishes to sell multiple homogeneous assets is permitted to decide at each point in time up to the deadline between 1) proposing a selling price up front to an appearing buyer or 2) concealing the price and letting the buyer come up with an offer. Our analysis indicates that under certain conditions there emerges a time threshold after which the seller switches from concealing his idea for the selling price to proposing this price, or vice versa. In addition, our numerical study also suggests that the maximum total expected profit obtained from the problem with switching strategy may improve substantially upon that of the problem without switching strategy, and this justifies the adoption of switching strategy in the business of selling assets.

Keywords: Dynamic programming; Posted price mechanism; Reservation price mechanism

1 Introduction

There are many ways in which an asset can be sold in order to maximize the expected profit gained from selling it. According to Arnold and Lippman [1] there exist four major selling mechanisms: posted price, reservation price (sequential search), auction, and bargaining. As the meaning of the latter two mechanisms, auction and bargaining, are apparent, we shall provide a standard definition for the first two as stated in [3]. The posted price mechanism means that a seller proposes a selling price to each appearing buyer, who, judging from the proposed price, decides whether or not to purchase the asset. On the other hand, the reservation price mechanism assumes that the seller conceals the selling price and the appearing buyer offers a price. The seller then decides whether or not to sell the asset by comparing the offered price to his reservation price. A well known example of this mechanism is the Name-Your-Own-Price system.

These four mechanisms have attracted considerable attention with the researchers and practitioners in the economics and operations research communities over the years. The literature on them is extensive and well established. Also, an increasing number of researches examining the coexistence of two or more mechanisms have been conducted in an attempt to determine which of these mechanisms should be used [1] [2] [4] [14] [15]. A common feature of this work undertaken so far assumes that a single mechanism is employed throughout the entire planning horizon. None of this work considers the possibility of switching between mechanisms during the planning horizon. In our paper we propose a new selling strategy called the *switching strategy* where the seller is permitted to switch between the two selling mechanisms, posted price and reservation

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price mechanisms, over the planning horizon.

Let us convey the flavor of our proposed switching strategy with the following example. Suppose a seller has an asset, say a car that he must sell before a deadline for some reasons. If the car remains unsold at the deadline, he will have to sell it to a used-car dealer for a giveaway price or dispose of it by paying some cost. At any time before the deadline, the seller has three possible options: To employ the posted price mechanism throughout the planning horizon, to employ the reservation price mechanism throughout the planning horizon, or to switch between employing the posted price (propose a price) and reservation price (conceal the price) mechanisms at certain points in time during the planning horizon. He then has to make a decision as to which one to choose out of the three options. To clarify the distinction between strategy with switching and that without switching, we shall call the strategy stated in the first and second options as *non-switching strategy.*

The asset selling problem with switching strategy proposed in this paper builds on the literature on the sequential search and posted price problem. The reservation price mechanism considered in our model is closely related to the works on sequential search which aims to maximize the expected value of an offer that will be accepted from the search process [9] [10] [11] [13]. The literature on the posted price problem can be separated into two categories; the single fixed pricing problem and dynamic pricing problem. In the first category it is assumed that a single fixed price is offered by the seller and all arriving buyers will purchase the asset at this price [12] [16] [18]. On the other hand, the literature in the second category assumes that selling prices can be dynamically adjusted at each point in time over the entire planning horizon. The objective of the literature belonging to this category is to determine the optimal pricing policy to implement at each point in time. A thorough review of the research on the dynamic pricing problem is given in [5] and [7]. The posted price mechanism considered in our model belongs to the second category.

In addition, our paper is also related to the articles on the selling mechanism selection problem [1] [2] [14] [15]. In these articles, separate models are developed for different selling mechanisms and they are compared against each other to determine which mechanism to choose. These articles show that there may exist a threshold value on which the optimal mechanism is selected. In this case, once a mechanism is committed to, there will be no mechanism change so long as the selling process proceeds. For example, Arnold and Lippman [1] shows that in the multiple homogeneous assets selling problem, if the number of items on hand is less than a threshold value, the seller should employ the reservation price mechanism, or else employ the auction mechanism. Let us pause to note the following. In the above mentioned example, suppose auction is selected because the quantity of items on hand at the start of the process is greater than the threshold value. A switch in the mechanism from auction to reservation price will not occur even if the quantity of items falls below the threshold value due to some items being sold out as the selling process proceeds. To the best of our knowledge, no model with the introduction of the switching strategy has ever been proposed so far. The main purpose of this paper is to propose a model for the asset selling problem in which the switching strategy is taken into consideration and to clarify the conditions for which switching is optimal (see Theorem 6.2).

In order to demonstrate that the switching strategy might prove beneficial, in this paper we assess the economic effectiveness of adopting the switching strategy by numerically comparing the maximum total expected profits for the model with switching strategy to that for the model with non-switching strategy (posted price or reservation price). For convenience, in the remainder of the paper we shall call the model with switching strategy the *switching model*, and the one with non-switching strategy the *non-switching model*. The results of our experiments show that the proposed switching model may improve substantially upon the non-switching models. The switching model might be thought to be an imaginary model; however, if the relative difference in the maximum expected profits between the switching and non-switching models is large enough not to be negligible, the seller would incur an opportunity loss for not adopting the switching strategy. In fact, we demonstrate by numerical examples in Section 7.2 that the relative difference may be greater than 20 percent. This fact, from a practical viewpoint, justifies the adoption of the switching strategy in the process of selling assets, provided that its adoption is feasible. Fortunately, recent advances in information technology such as internet-based selling systems will make such adoption easy.

The rest of this paper is organized as follows. Section 2, that follows, provides a strict definition of our model. Section 3 defines several functions and examines their properties, as they are used in the subsequent analysis. In Section 4 we derive the optimal equations for the model, and in Section 5 we state the optimal decision rules. In Section 6 we clarify the properties of the optimal decision rules. In Section 7 we provide numerical examples that ascertain the existence of switching property and examine the economic effectiveness of adopting switching strategy. Finally, in Section 8 we present the overall conclusions of our research and suggest some further work which could be done.

2 Model

The model for the dynamic asset selling process discussed in this paper is defined on the assumptions stated below:

1. Consider the following discrete-time sequential stochastic decision problem of selling multiple homogeneous items within a finite planning horizon. The points in time are numbered backward from the final point in time of the planning horizon, time 0 (the deadline) as $0, 1, \cdots$ and so on. Accordingly, if time t is the present point in time, the two adjacent times $t+1$ and $t-1$ are the previous and next points in time, respectively. Let the time interval between times t and $t-1$ be called the *period* t, which is small enough so that no more than one buyer may appear. Each appearing buyer is assumed to require no more than one item. Furthermore, we do not consider the discount factor and the holding cost of storing the unsold items. We refer the interested reader to [6] for the analysis of the switching model with discount factor and holding cost.
2. An item remaining unsold at time 0, the deadline, can be sold at a finite salvage price $\rho \in (-\infty, \infty)$. Here, $\rho < 0$ implies the disposal cost per item to discard an unsold item.
3. A buyer who requests an item arrives with a probability λ $(0 < \lambda < 1)$.

4. When a buyer appears, the seller has to make a decision between two alternatives: $A1$ ⟨proposing a selling price⟩ and $A2$ ⟨concealing the selling price⟩.

A1. If the seller chooses the first alternative, the buyer then decides whether or not to purchase the item based on the price offered by the seller. By ξ let us denote the maximum permissible buying price of a buyer, implying that the buyer is willing to purchase if and only if the selling price z proposed by the seller is lower than or equal to ξ, i.e., $z \le \xi$. Here, assume that subsequent buyers' maximum permissible buying prices $\xi,\ \xi', \cdots$ are independent identically distributed random variables having a known continuous distribution function $F(\xi)$ with a finite expectation μ; let $f(\xi)$ denote its probability density function, which is truncated on both sides. More precisely, $F(\xi)$ and $f(\xi)$ are defined as follows. For certain given numbers a and b such that $0 < a < b < \infty$

$$F(\xi) = 0, \quad \xi \le a, \qquad 0 < F(\xi) < 1, \quad a < \xi < b, \qquad F(\xi) = 1, \quad b \le \xi, \tag{2.1}$$

where

$$f(\xi) = 0, \quad \xi < a, \qquad f(\xi) > 0, \quad a \le \xi \le b, \qquad f(\xi) = 0, \quad b < \xi. \tag{2.2}$$

Then clearly $a < \mu < b$. Thus, the probability of an appearing buyer purchasing the asset, provided that a price z is offered by the seller, is given by $p(z) = \Pr\{z \le \xi\}$ where $0 \le p(z) \le 1$. Then it can be easily seen that

$$p(z) \begin{cases} = 1, & z \le a, \cdots (1), \\ < 1, & a < z, \cdots (2), \end{cases} \qquad p(z) \begin{cases} > 0, & z < b, \cdots (3), \\ = 0, & b \le z, \cdots (4). \end{cases} \tag{2.3}$$

Furthermore, let us define

$$\underline{f} = \inf\{f(\xi) \mid \xi \in [a, b]\} > 0, \tag{2.4}$$

which will become inevitably necessary to successfully prove Lemma 3.3(a).

A2. If the seller chooses the second alternative, the buyer will definitely offer a price. The seller will then decide whether or not to sell the item judging from the price offered by the buyer. In this case, it is assumed that the buyer offers a price $w = \alpha\xi$ $(0 < \alpha \le 1)$ where ξ is the buyer's maximum permissible buying price. Here, let us call the α the *price offering ratio*, which measures the degree of a buyer's desirability for the asset; the greater (lower) the buyer's desirability may be, the closer the α may be to 1 (0). In this paper, we assume that α and ξ are stochastically independent and that subsequent buyers' price offering ratios, $\alpha, \alpha', \cdots$, are independent identically distributed random variables having a known distribution function $F_0(\alpha)$ with a finite expectation $\mu_0 > 0$, i.e., $\mathbf{E}_\alpha[\alpha] = \mu_0$. Let $F_1(w)$ and $f_1(w)$ denote, respectively, the distribution function of w and its probability density function; by μ_1 let us represent the expectation of w. Then clearly

$$\mu_1 = \mathbf{E}[w] = \mathbf{E}[\alpha\xi] = \mu_0\mu > 0. \tag{2.5}$$

Hence we have $F_1(x) = \Pr\{w \le x\} = \Pr\{\alpha\xi \le x\} = \Pr\{\xi \le x/\alpha\} = \mathbf{E}_\alpha[F(x/\alpha)]$. Accordingly, the probability density function $f_1(x)$ is given by

$$f_1(x) = \mathop{\mathbf{E}}_{\alpha}[1/\alpha f(x/\alpha)]. \tag{2.6}$$

The decision rules of the model consist of:

1. The *Switching rule* as to when to switch the alternative to be taken from A1 to A2 or from A2 to A1.
2. The *Pricing rule* as to what price to offer to an arriving buyer when the seller takes alternative A1.
3. The *Selling rule* as to whether to sell the item or not when the seller takes alternative A2.

The objective here is to find the optimal decision rules so as to maximize the total expected profit over the planning horizon, i.e., the total expected revenue gained from selling the items to appearing buyers *plus* the total expected salvage value of the items remaining unsold at the deadline.

3 Preliminaries

This section defines the functions that will be used to describe the optimal equations of the model (see Section 4). The properties of the functions verified in this section will be applied to the analysis of our model. First, for any x let us define the following two functions:

$$T_1(x) = \int_0^\infty \max\{\xi - x, 0\} f(\xi) d\xi, \tag{3.1}$$

$$T_c(x) = \int_0^\infty \max\{w - x, 0\} f_1(w) dw. \tag{3.2}$$

Rearranging Eq. (3.2) by substituting Eq. (2.6) yields $T_c(x) = \mathbf{E}_\alpha[1/\alpha \int_0^\infty \max\{w-x, 0\} f(w/\alpha) dw]$. Then since $w = \alpha\xi$ by definition, noting Eq. (3.1), we get

$$T_c(x) = \mathbf{E}_\alpha[\alpha \int_0^\infty \max\{\xi - x/\alpha, 0\} f(\xi) d\xi] = \mathbf{E}_\alpha[\alpha T_1(x/\alpha)]. \tag{3.3}$$

Next, for any real number x let us define

$$T_p(x) = \max_z p(z)(z - x), \tag{3.4}$$

and by $z(x)$ let us designate the smallest z attaining the maximum of the right-hand side of Eq. (3.4) if it exists, i.e.,

$$T_p(x) = p(z(x))(z(x) - x). \tag{3.5}$$

The function $T_p(x)$ is also defined in [17], which studied the asset selling problem using the posted price mechanism where a buyer can only be found by paying a search cost. Here, let us introduce two properties of $z(x)$ whose proofs can be found in [17].

Lemma 3.1 (You [17])

(a) *$z(x)$ is nondecreasing in $x \in (-\infty, \infty)$ with $z(x) \geq a$ for any x.*

(b) *If $x \geq (<)\ b$, then $z(x) = b$ $(x < z(x) < b)$.*

As will be shown in Section 4, the decision is made between the alternatives A1 and A2 by comparing the values of the functions $T_c(x)$ and $T_p(x)$. For this reason, let us define

$$J(x) = T_{\mathrm{c}}(x) - T_{\mathrm{p}}(x), \tag{3.6}$$

$$\mathcal{T}(x) = \max\{T_{\mathrm{c}}(x), T_{\mathrm{p}}(x)\} = \max\{J(x), 0\} + T_{\mathrm{p}}(x). \tag{3.7}$$

Furthermore, we shall define

$$a^\star = \inf\{x \,|\, T_{\mathrm{p}}(x) > a - x\}, \qquad a^\circ = \max\{x \,|\, T_{\mathrm{c}}(x) = \mu_1 - x\}, \tag{3.8}$$

$$x^\star = \inf\{x \,|\, z(x) > a\}, \qquad b^\circ = \sup\{x \,|\, T_{\mathrm{c}}(x) > 0\}, \qquad c^\circ = \min\{x^\star, a^\circ\}, \tag{3.9}$$

if they exist. The above defined five symbols and their relationships are needed in examining the properties of $\mathcal{T}(x)$ and $J(x)$, which are summarized in the two lemmas below.

Lemma 3.2

(a) *$\mathcal{T}(x)$ is continuous and nonincreasing on $(-\infty, \infty)$.*

(b) *$\mathcal{T}(x) > 0$ on $(-\infty, b)$ and $\mathcal{T}(x) = 0$ on $[b, \infty)$.*

(c) *$\lambda \mathcal{T}(x) + x$ is strictly increasing on $(-\infty, \infty)$.*

(d) *If $x \le (\ge)\ y$, then $0 \le (\ge)\ \mathcal{T}(x) - \mathcal{T}(y) \le (\ge)\ (y - x)$.*

Proof. See Appendix A. ∎

Lemma 3.3

(a) *$x^\star \le a^\star < a$ and $c^\circ \le b^\circ \le b$.*

(b) *$J(x) = \mu_1 - a$ on $(-\infty, c^\circ]$ and $J(x) = 0$ on $[b, \infty)$.*

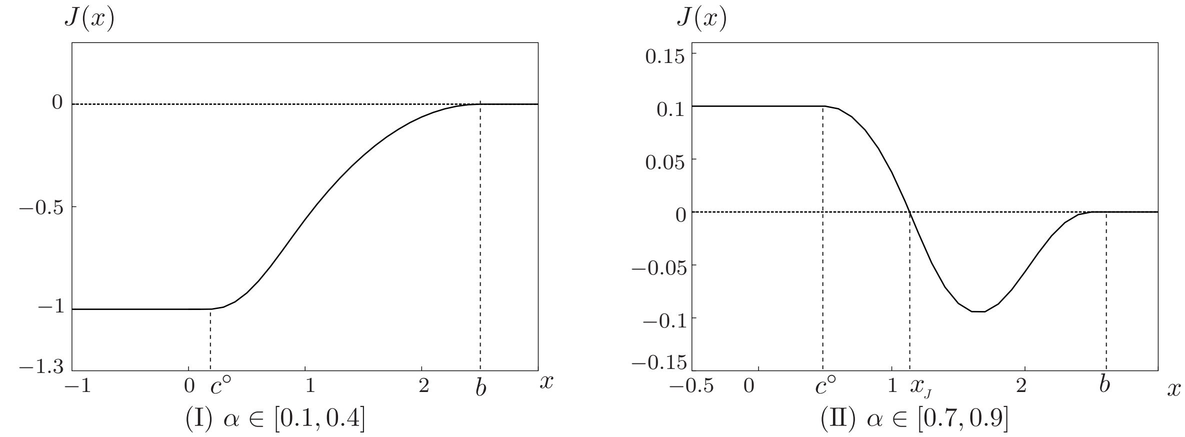
(c) *If $b^\circ < b$, then $J(x)$ is strictly increasing and negative (i.e., $J(x) < 0$) on $[b^\circ, b)$.*

Proof. See Appendix B. ∎

From Lemma 3.3 we can only partially specify the shape of the function $J(x)$, that is $J(x) = \mu_1 - a$ on $(-\infty, c^\circ]$ and $J(x) = 0$ on $[b, \infty)$, in other words, $J(x)$ is constant on $(-\infty, c^\circ]$ and $[b, \infty)$. However, its shape on (c°, b) cannot be easily determined. It will be seen later that the shape of this function decisively influences on whether or not the optimal decision rules exhibit the existence of switching at some points during the planning horizon.

Now, in general by x_{J} let us denote the solutions of the equation $J(x) = 0$ if they exist, i.e., $J(x_{J}) = 0$. We will see later on that only the solutions on the interval (c°, b) characterize the properties of the optimal decision rules. Noting these facts, let us provide a more precise definition of the solution as follows. For certain α and β such that $c^\circ < \alpha \le \beta < b$, if $J(x) = 0$ on $[\alpha, \beta]$ with $J(\alpha - \varepsilon) \ne 0$ and $J(\beta + \varepsilon') \ne 0$ for any infinitesimal $\varepsilon > 0$ and $\varepsilon' > 0$, then let $x_{J} = \alpha$. If $\alpha = \beta$, the solution is the *isolated* solution. Note that the solution x_{J} defined in this matter may be multiple, such that $c^\circ < x_{J}^1 < \cdots < x_{J}^k < b$ for $1 \le k \le N$. Below let us give some examples showing that $J(x) = 0$ may or may not have isolated solution on the interval (c°, b) depending on the given distribution functions $F(\xi)$ and $F_0(\alpha)$.

Example 1 Let $F(\xi)$ be the uniform distribution function on $[1.5, 2.5]$. If $F_0(\alpha)$ is a uniform distribution function on $[0.1, 0.4]$, then $J(x) = 0$ has no solution on the interval (c°, b) with



(I) $\alpha \in [0.1, 0.4]$ (II) $\alpha \in [0.7, 0.9]$

Figure 3.1: The shapes of $J(x)$

$c^\circ \approx 0.1500$ (see Figure 3.1(I)), and if $F_0(\alpha)$ is a uniform distribution function on $[0.7, 0.9]$, then $J(x) = 0$ has an isolated solution, $x_J \approx 1.1339$ on the interval (c°, b) with $c^\circ \approx 0.50$ (see Figure 3.1(II)).

Example 2 Consider $F(\xi)$ with $f(\xi)$ such that $f(\xi) \approx 0.05701$ on $[0.1, 0.599]$, $f(\xi)$ is a triangle on $[0.599, 0.7]$ with its maximum at $\xi = 0.6$, and $f(\xi) \approx 0.06982$ on $[0.7, 3.0]$. Let $F_0(\alpha)$ be a uniform distribution on $[0.64, 0.74]$. Then $J(x) = 0$ has three isolated solutions, $x_J^1 \approx -0.5566$, $x_J^2 \approx 0.4630$, and $x_J^3 \approx 0.7471$, on the interval (c°, b) with $c^\circ \approx -17.8272$ as shown in Figure 3.2(II).

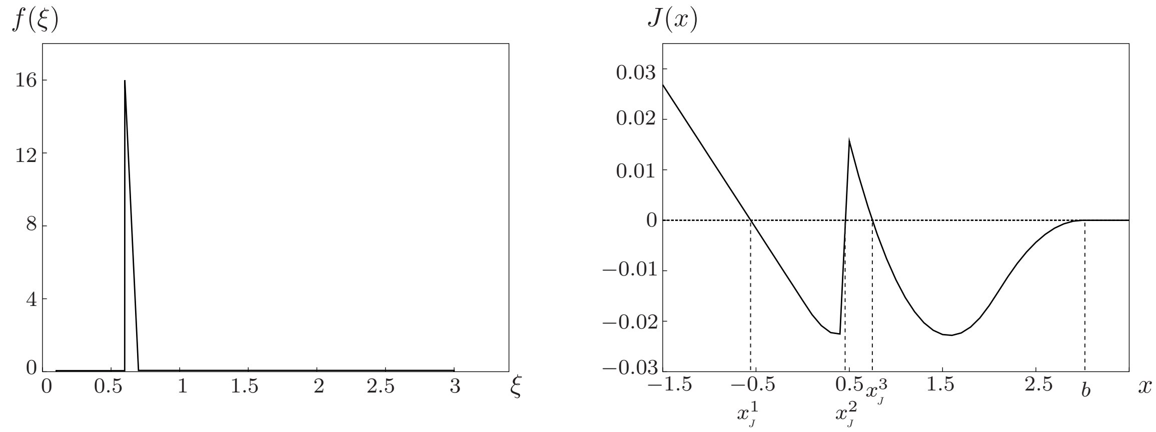


Figure 3.2: $f(\xi)$ and $J(x)$ where $c^\circ \approx -17.8272$

4 Optimal Equations

Suppose that a certain number of items has been purchased at a certain past point in time and that i items remain unsold at a time t after that. Let $u_t(i, \phi)$ and $u_t(i, 1)$ be the maximum total expected profits, respectively, with no buyer and with a buyer. Then, clearly

$$u_0(i, \phi) = \rho i, \qquad u_t(0, \phi) = u_t(0, 1) = 0, \quad t \geq 0, \quad i \geq 0, \tag{4.1}$$

and

$$u_t(i,\phi) = \lambda u_{t-1}(i,1) + (1-\lambda)u_{t-1}(i,\phi), \qquad t \geq 1, \quad i \geq 0, \tag{4.2}$$

$$u_t(i,1) = \max \left\{ \begin{array}{ll} \text{A1}: \max_z \{p(z)\big(z + u_t(i-1,\phi)\big) + (1-p(z))u_t(i,\phi)\} & \cdots (1), \\ \text{A2}: \displaystyle\int_0^\infty \max\{w + u_t(i-1,\phi), u_t(i,\phi)\} f_1(w)dw & \cdots (2), \end{array} \right\}, \quad t \geq 0, \quad i \geq 1. \tag{4.3}$$

Below, for convenience of the later analysis, let us transform the above equations. First, define

$$U_t(i) = u_t(i,\phi) - u_t(i-1,\phi), \qquad t \geq 0, \quad i \geq 1, \tag{4.4}$$

and let $U(i) = \lim_{t\to\infty} U_t(i)$ for any given $i \geq 1$ if it exists. From Eq. (4.1) we have

$$U_0(i) = \rho, \qquad i \geq 1. \tag{4.5}$$

Now, from Eqs. (3.2) and (3.4) we obtain, respectively, $\int_0^\infty \max\{w,x\} f_1(w)dw = T_\mathrm{c}(x) + x$ and $\max_z\{p(z)z + (1-p(z))x\} = T_\mathrm{p}(x) + x$. Then noting the function $\mathcal{T}(x)$ defined by Eq. (3.7), we can rewrite Eq. (4.3) as follows.

$$\begin{aligned} u_t(i,1) &= \mathcal{T}(U_t(i)) + u_t(i,\phi) & (4.6) \\ &= \max\{J(U_t(i)), 0\} + T_\mathrm{p}(U_t(i)) + u_t(i,\phi), \qquad t \geq 0, \quad i \geq 1. & (4.7) \end{aligned}$$

For convenience, let

$$U_t(0) = M, \qquad t \geq 0, \tag{4.8}$$

for a sufficiently large M such that $\rho < M$ and $b < M$. Then since $\mathcal{T}(U_t(0)) + u_t(0,\phi) = \mathcal{T}(M) = 0 = u_t(0,1)$ due to Eq. (4.1) and Lemma 3.2(b), we see that Eq. (4.6) holds for $i \geq 0$ instead of $i \geq 1$. Accordingly, owing to the fact that $u_{t-1}(i,1) - u_{t-1}(i,\phi) = \mathcal{T}(U_{t-1}(i))$ for $t \geq 1$ and $i \geq 0$ from Eq. (4.6), we can rewrite Eq. (4.2) as

$$u_t(i,\phi) = \lambda \mathcal{T}(U_{t-1}(i)) + u_{t-1}(i,\phi), \qquad t \geq 1, \quad i \geq 0, \tag{4.9}$$

from which we get

$$U_t(i) = \lambda\big(\mathcal{T}(U_{t-1}(i)) - \mathcal{T}(U_{t-1}(i-1))\big) + U_{t-1}(i), \qquad t \geq 1, \quad i \geq 1. \tag{4.10}$$

5 Optimal Decision Rules

From Eq. (4.7) the optimal decision rules for a given $t \geq 0$ and $i \geq 1$ can be prescribed as follows.

(a) If $J(U_t(i)) \geq 0$, conceal the selling price. Then, for a price w offered by a buyer appearing at that time, if $w \geq U_t(i)$, sell the item, or else do not; in other words, $U_t(i)$ becomes the seller's *minimum permissible selling price* [‡].

(b) If $J(U_t(i)) \leq 0$, offer a price to an appearing buyer. The optimal selling price for an item remaining unsold at that time is given by the smallest z attaining the maximum of Eq. (4.3 (1))

[‡] Minimum permissible selling price means the reservation price of the seller for selling the asset; he is willing to sell the asset if and only if the price offered by the buyer is greater than his reservation price.

if it exists, denoted by $z_t(i)$. Since Eq. (4.3 (1)) can be expressed as $T_{\mathrm{p}}(U_t(i)) + u_t(i, \phi) = \max_z p(z)(z - U_t(i)) + u_t(i, \phi)$, we have $z_t(i) = z(U_t(i))$ due to the definition of $z(x)$ (see Eq. (3.5)).

6 Analysis

This section is devoted to examining the properties of the optimal decision rules. To do so, first we need the following lemma.

Lemma 6.1

(a) *$U_t(i)$ is nonincreasing in $i \geq 0$ for $t \geq 0$ and nondecreasing in $t \geq 0$ for $i \geq 0$.*

(b) *$U_t(i)$ converges to $U(i)$ as $t \to \infty$ for $i \geq 1$ with $U(i) \geq b$.*

(c) *If $\rho < b$, then for $i \geq 1$ we have $U_t(i) < b$ for $t \geq 0$ and $U(i) = b$.*

Proof. See Appendix C. ∎

Theorem 6.1 *The optimal selling price $z_t(i)$ is nonincreasing in $i \geq 0$ for $t \geq 0$ and nondecreasing in $t \geq 0$ for $i \geq 0$ with $a \leq z_t(i) \leq b$.*

Proof. Evident from the facts that $z_t(i) = z(U_t(i))$ by definition and that the monotonicity of $U_t(i)$ in i and t is inherited to $z_t(i)$ due to Lemma 3.1(a); Lemmas 6.1(a); and 3.1. ∎

Intuition suggests the following. If the seller has substantial items remaining unsold at a point in time or if the deadline is approaching, in order to avoid leftover items at the deadline, he may become more compelled to sell, implying that he will lower the selling price (if proposing a price is optimal) or his minimum permissible selling price (if concealing the price is optimal) as the number of items remaining unsold i increases or as the remaining time periods up to the deadline t decreases. Therefore, it can be conjectured that $z_t(i)$ and $U_t(i)$ are both nonincreasing in i and nondecreasing in t. Lemma 6.1(a) and Theorem 6.1 affirm our conjecture; these results are consistent with those in [1], [3], [8], and [17]. Furthermore, another immediate consequence of Theorem 6.1 is that if the alternative A1 is optimal, the seller will charge a price which lies in between a and b, the upper and lower bounds of the distribution function of the buyer's reservation price.

Below, we shall provide the strict definitions of the switching property.

Definition 6.1 *Let $i \geq 1$.*

(a) *When t moves from 0 to ∞, if $J(U_t(i))$ changes from < 0 to > 0 or from > 0 to < 0, let the sign of $J(U_t(i))$ be said to change; If the sign change occurs, the optimal decision rules have a switching property, or else do not.*

(b) *If a sign change occurs $k \geq 0$ times, let the optimal decision rules be said to possess the k-switching property.*

(c) *Let us refer to the point in time when the sign change of $J(U_t(i))$ occurs as the switching time threshold, denoted by $t^*(i)$.*

The $k = 0$ implies that the optimal decision rules have *no* switching property. For explanatory convenience, if $k = 1$ ($k \geq 2$), it is said to possess a *single* (*multiple*) switching property. As time t moves from 0 to ∞, $U_t(i)$ starting with $U_0(i)$ (equivalent to ρ due to Eq. (4.5)) increases and converges to $U(i) \geq b$ (see Lemma 6.1(b)). Paying attention to this fact, we obtain the following theorem which prescribes whether sign change of $J(U_t(i))$ occurs, and from this it will be known whether or not the optimal decision rules possess the switching property as the process proceeds.

Theorem 6.2

(a) *Let $b \leq \rho$. Then the optimal decision rules have no switching property.*

(b) *Let $\rho < b$.*

1 *Let $J(x) = 0$ have no solution x_J on (c°, b). Then the optimal decision rules have no switching property.*

2 *Let $J(x) = 0$ have $k \geq 1$ solutions x_J^1, x_J^2, ... , x_J^k on (c°, b) such that $x_J^1 < x_J^2 < \ldots < x_J^k$. Further, let $x_J^\ell \leq \rho < x_J^{\ell+1}$ for a given ℓ such that $0 \leq \ell \leq k$ where $x_J^0 = -\infty$ and $x_J^{k+1} = \infty$. Then the optimal decision rules have the $(k - \ell)$-switching property.*

Proof. Note that $U_0(i) = \rho$ from Eq. (4.5) and that $U_t(i)$ is monotone in t from Lemma 6.1(a).

(a) Let $b \leq \rho$. Then since $b \leq U_0(i) \leq U_t(i)$ for $t \geq 0$, we have $J(U_t(i)) = 0$ for $t \geq 0$ from Lemma 3.3(b). Hence in this case, sign change of $J(U_t(i))$ does not occur on $t \geq 0$, so the optimal decision rules have no switching property.

(b) Let $\rho < b$.

(b1) Immediate from the fact that $J(x)$ is constant on $(-\infty, c^\circ]$ and $[b, \infty)$ from Lemma 3.3(b).

(b2) If $x_J^{\ell+1}$, $x_J^{\ell+2}$, ... , x_J^k are isolated solutions, the sign of $J(U_t(i))$ changes at these solutions as t moves from 0 to ∞. Thus the assertion holds. Even if all of $x_J^{\ell+1}$, $x_J^{\ell+2}$, ... , x_J^k are not isolated solutions, clearly sign change of $J(U_t(i))$ also occurs k times as t moves from 0 to ∞. Consequently, the assertion holds. ∎

7 Numerical Experiments

The objective of the numerical experiments is twofold: to exemplify the existence of the switching property and to assess the economic effectiveness of adopting the switching strategy.

7.1 Switching property

The optimal decision rules in Theorem 6.2 are prescribed on the assumption that none of the $i \geq 1$ items on hand at the starting point of the process will be sold out throughout the entire planning horizon, i.e., all the items will remain unsold up to the deadline. Here, we provide some examples where the optimal decision rules have 0, single, or multiple switching properties. Consider $f(\xi)$ and $F_0(\alpha)$ defined in *Example 2* where $b = 3.0$. Then $J(x)$ can be depicted as in Figure 7.3. Here, let $\lambda = 0.5$ and $i = 5$. Then by calculation we obtain $c^\circ \approx -17.8272$ and know that $J(x) = 0$ has three isolated solutions: $x_J^1 \approx -0.5566$, $x_J^2 \approx 0.4630$, and $x_J^3 \approx 0.7471$, i.e., $k = 3$ and $x_J^1 < x_J^2 < x_J^3$. Let $\rho = -3.0$, hence $\rho < b$, satisfying the condition of Theorem 6.2(b). In this case, since $\rho = -3.0 < -0.5566 \approx x_J^1$, the condition in Theorem 6.2(b2) with $\ell = 0$ is also satisfied. Consequently, it follows that the optimal decision rules have a 3-switching property for

$t \geq 0$. Similarly, if $\rho = 0.01$ and 0.5, then it can be easily confirmed that the optimal decision rules have, respectively, a 2-switching property and a 1-switching property. Finally, if $\rho = 1.5$, the optimal decision rules have 0-switching property.

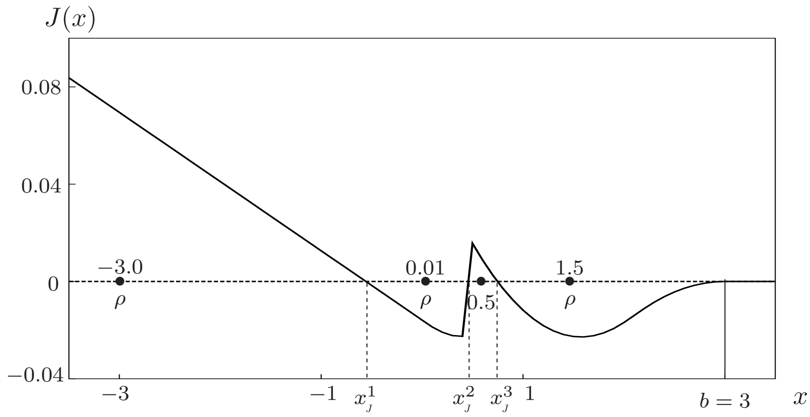


Figure 7.3: Switching property where $c^\circ \approx -17.8272$.

In reality, the assumption stated above may fail to hold since it is possible that some items are sold to appearing buyers as the process proceeds. Taken this fact into consideration, we need to interpret the switching property as illustrated in the following two scenarios. Let $F(\xi)$ and $F_0(\alpha)$ be the uniform distribution functions, respectively, on $[1.5, 15.5]$ and $[0.58, 0.9]$; and let $\lambda = 0.54$ and $\rho = 0.7$. From the calculation we have $t^*(1) = 2$, $t^*(2) = 5$, and $t^*(3) = 9$. Here note that for each of $i = 1$, 2, and 3 the seller should conceal the selling price if $t \leq t^*(i)$, or else propose the price. In the two scenarios below let the process start from $t = 11$ when a seller has $i = 3$ items on hand (see Figure 7.4).

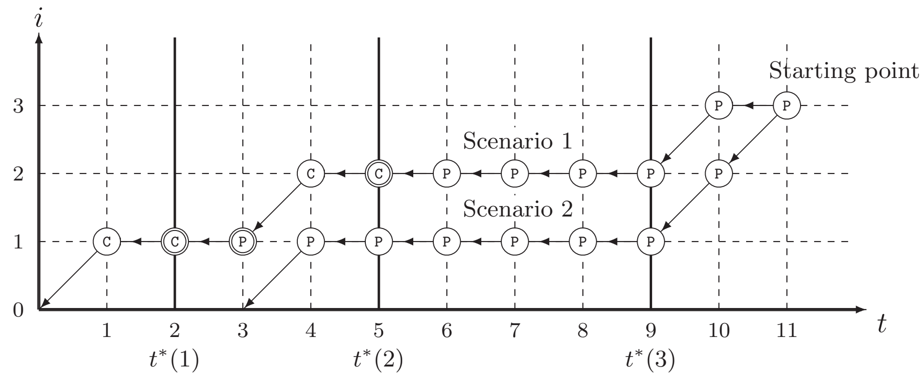


Figure 7.4: Scenarios of selling process (The symbols ⓒ and ⓟ represent the decisions of, respectively, concealing and proposing the selling price, and the symbol ◎ indicates that switching occurs at that time point).

Scenario 1 Since $t = 11 > 9 = t^*(3)$, the seller should propose a price. If no item is sold at that time, the process proceeds to $t = 10$ and the seller should propose a price at that time since $t = 10 > 9 = t^*(3)$. Assume that one item is sold at $t = 10$, hence the number of items on hand at $t = 9$ is reduced to $i = 2$. Then the seller should propose a price at that time since

$t = 9 > 5 = t^*(2)$. If no item is sold at $t = 9$, the process proceeds to $t = 8$ and the seller should propose a price since $t = 8 > 5 = t^*(2)$. Similarly, if no item is sold up to $t = 6$, the seller should propose a price at that time since $t = 6 > 5 = t^*(2)$. Since $t = 5 = t^*(2)$, the seller should *switch* to concealing the price. If no item is sold at that time, the process proceeds to $t = 4$ and the seller should conceal the price at that time since $t = 4 < 5 = t^*(2)$. Assume that one item is sold at that time. Then the number of items at $t = 3$ is reduced to $i = 1$, hence the seller should *switch* to proposing the price at that time since $t = 3 > 2 = t^*(1)$. If no item is sold at $t = 3$, the process proceeds to $t = 2$ and the seller should *switch* to concealing the price since $t = t^*(1) = 2$. Accordingly, in this scenario, switching occurs three times throughout the entire planning horizon starting from time $t = 11$.

Scenario 2 Since $t = 11 > 9 = t^*(3)$, the seller should propose a price. Assume that one item is sold at that time, so the number of items at $t = 10$ is reduced to $i = 2$, hence the seller should propose a price at that time since $t = 10 > 5 = t^*(2)$. Assume that one item is sold at that time, hence the number of items at $t = 9$ is reduced to $i = 1$. Then the seller should propose a price at that time since $t = 9 > 2 = t^*(1)$. If no item is sold at that time, the process proceeds to $t = 8$ and the seller should propose a price at that time since $t = 8 > 2 = t^*(1)$. Similarly, if no item is sold up to $t = 4$, the seller should propose a price at that time since $t = 4 > 2 = t^*(1)$. Suppose that the last item is sold at $t = 4$. Then in this scenario, it eventually follows that no switch of action occurs throughout the entire planning horizon.

We should notice that scenario 2 exemplifies that although switching time thresholds exist for each of $i = 1, 2$, and 3 (i.e., switching occurs if i items on hand at the starting point remain unsold up to the deadline), switching might not occur if i items gradually decrease due to being sold out as the selling process proceeds.

7.2 Economic effectiveness of adopting switching strategy

First, let $\tilde{u}_t(i, \phi)$ be the maximum total expected profit of non-switching model which employs the non-switching strategy, posted price or reservation price, for $t \geq 0$ and $i \geq 0$, provided that no buyer exists. Then let $\varphi_t(i)$ be the *relative difference* in the maximum total expected profits between the switching and non-switching models, provided that $u_t(i, \phi) \neq 0$, i.e., $\varphi_t(i) = (u_t(i, \phi) - \tilde{u}_t(i, \phi))/u_t(i, \phi)$. For convenience, let $\varphi_t(i)$ for posted price and reservation price mechanism be denoted by, respectively, $\varphi_t^p(i)$ and $\varphi_t^c(i)$. For all the numerical results illustrated below, let $F(\xi)$ and $F_0(\alpha)$ be the uniform distribution functions, respectively, on $[1.5, 15.5]$ and $[0.58, 0.9]$; and let $\lambda = 0.54$ and $\rho = 0.7$. Our main observations are the following:

1) Figure 7.5(I,II) depicts the monotonicity of, respectively, $\varphi_t(1)$ and $\varphi_t(10)$ in t. In this case we have the switching time threshold $t^*(1) = 2$ and $t^*(10) = 35$. When $i = 1$ ($i = 10$), it is optimal to conceal a price if $t \leq 2$ ($t \leq 35$) and propose it if $t > 2$ ($t > 35$). In addition, from Figure 7.5(I) we see that $\varphi_t^p(1)$ and $\varphi_t^c(1)$ can be as high as 24% and 16%, respectively. This implies that the seller may bear the risk of having a large difference in the maximum total expected profits by using non-switching strategy (either posted price or reservation price mechanism) throughout the entire planning horizon, instead of employing the switching strategy.

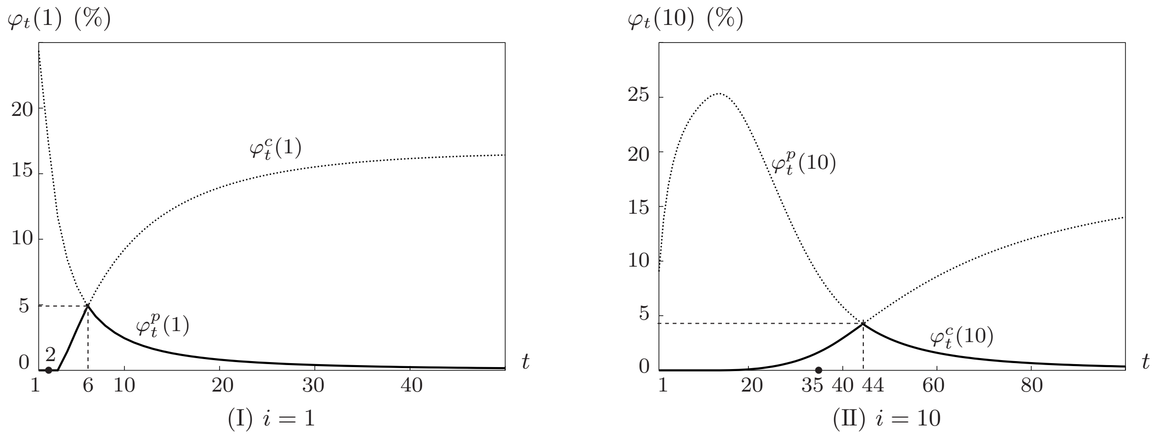


Figure 7.5: Relationship of $\varphi_t^c(i)$ and $\varphi_t^p(i)$ with t where the symbol $\bullet$ indicates the switching time threshold $t^*(i)$.

2) As illustrated in Section 7.1, a proper adoption of switching strategy in business requires the monitoring of the quantity on hand. Thus if the relative difference between the switching and non-switching models is not large enough, the sellers may confront the situation where the cost of frequent monitoring exceeds the benefit for adopting the switching strategy. In such case, from a practical viewpoint, the seller should not employ the switching strategy. Suppose for this reason, the seller determines to employ the non-switching strategy. Then the bold curves of Figure 7.5(I,II) tell us the following. When $i = 1$ ($i = 10$) it is optimal to adopt the reservation price mechanism throughout the entire planning horizon if the process starts from time $t \leq 6$ ($t \leq 44$) and the posted price mechanism if from $t > 6$ ($t > 44$). In addition, it should be noted that when $i = 1$ ($i = 10$) the maximum relative difference for employing non-switching strategy is approximately 5% (4.3%), which occurs at $t = 6$ ($t = 44$), and that the relative difference decreases gradually as the planning horizon becomes greater or less than 6 (44). Furthermore, we obtain a numerical example demonstrating that the maximum relative difference for employing non-switching strategy may increase more than twofold; for instance, it becomes 11% when $i = 1$, $\rho = -5.0$ and $F_0(\alpha)$ be uniform on $[0.3, 0.9]$ by letting λ and $F(\xi)$ unchanged.

8 Conclusions and Suggested Future Studies

In this paper we have proposed a basic model for an asset selling problem where the seller can switch between proposing a selling price to appearing buyers and concealing the price. From our analysis, we obtained some conditions that guarantee the existence of the switching property. Below, we shall reemphasize the two distinctive points derived from our analysis.

1. Through the analysis, we showed that the optimal decision rules may possess switching property; a numerical experiment also demonstrated that multiple switching property may exist.
2. The results of the numerical experiments we obtained in Section 7 demonstrated that the adoption of the switching strategy may be effective in increasing the seller's profit. We showed that the relative difference between the maximum total expected profits for the switching and

non-switching models can be as high as 24 percent. This implies that a seller may incur an opportunity loss if the switching strategy is not employed.

Below, let us mention some interesting directions for extending our model that could make it more practical. Possible extensions include: 1) consideration of the cost of monitoring the current quantity of items on hand and conducting mechanism switching, 2) future availability of the buyer who leaves the selling process without purchase, 3) proposed selling price affecting the buyer arriving probability λ, and 4) introduction of a cost of attracting buyers, called the search cost.

Appendix : Proofs

A. Proof of Lemma 3.2

Before proving this lemma we must introduce three additional lemmas and a proposition. The proof of Lemma A.1 can be found in [17].

Lemma A.1 (You [17])

(a) $T_{\rm p}(x)$ *is continuous*[§]*, nonincreasing on* $(-\infty, \infty)$*, and strictly decreasing on* $(-\infty, b)$*.*

(b) $T_{\rm p}(x) > 0$ *on* $(-\infty, b)$ *and* $T_{\rm p}(x) = 0$ *on* $[b, \infty)$*.*

(c) $T_{\rm p}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *and strictly increasing on* (a, ∞)*.*

(d) *If* $x \le (\ge)\ y$*, then* $0 \le (\ge)\ T_{\rm p}(x) - T_{\rm p}(y) \le (\ge)\ y - x$[¶]*.*

Proposition A.1 *Let a given continuous function* $g(x)$ *be nonincreasing (nondecreasing) on* $(-\infty, \infty)$*. Then if* $g(x)$ *is strictly decreasing (strictly increasing) on* $(-\infty, A)$ *or* (B, ∞) *for certain given finite* A *and* B*, so also is it on* $(-\infty, A]$ *or* $[B, \infty)$*.*

Proof. It suffices to show that if $x < A$, then $g(x) > (<)\ g(A)$, and if $x > B$, then $g(x) < (>)\ g(B)$. Assume $g(A) = g(x)$ for a certain $x < A$. Then $g(x') < (>)\, g(x) = g(A)$ for $x < x' < A$, which is a contradiction, hence it must be $g(A) < (>)\, g(x)$, implying that $g(x)$ is strictly decreasing (strictly increasing) on $(-\infty, A]$. It can be proven in a quite similar way that $g(x)$ must be strictly decreasing (strictly increasing) on $[B, \infty)$. ▮

Lemma A.2

(a) $T_1(x)$ *is continuous, nonincreasing, and convex in* $x \in (-\infty, \infty)$*.*

(b) $T_1(x) > 0$ *for* $x \in (-\infty, b)$ *and* $T_1(x) = 0$ *for* $x \in [b, \infty)$*.*

(c) $T_1(x) = \mu - x$ *for* $x \in (-\infty, a]$ *and* $T_1(x) > \mu - x$ *for* $x \in (a, \infty)$*.*

[§]The assertion that $T_{\rm p}(x)$ is continuous on $(-\infty, \infty)$ is not provided in [17]. However, this assertion is obvious from the fact that $p(z)(z - x)$ is continuous on $(-\infty, \infty)$ for any x.

[¶]Let $x \le (\ge)\ y$. Then $T_{\rm p}(x) - T_{\rm p}(y) \ge (\le)\ 0$ from (a). The assertion of $T_{\rm p}(x) - T_{\rm p}(y) \le y - x \cdots (1^*)$ if $x \le y$ is equivalent to the one of Lemma 3.1(e) in [17] since $T_{\rm p}(x) \ge 0$ from (b). Multiplying both sides of (1^*) by -1 leads to $T_{\rm p}(y) - T_{\rm p}(x) \ge x - y$ for $x \le y$, and then interchanging the notations x and y yields $T_{\rm p}(x) - T_{\rm p}(y) \ge y - x$ for $y \le x$.

Proof. $T_1(x) = \int_x^\infty (\xi - x)dF(\xi) \geq \int_y^\infty (\xi - x)dF(\xi)$ for any x and y, so

$$T_1(x) - T_1(y) \geq \int_y^\infty (\xi - x)dF(\xi) - \int_y^\infty (\xi - y)dF(\xi) = -(x - y)(1 - F(y)).$$

Similarly we get $T_1(x) - T_1(y) \leq -(x - y)(1 - F(x))$. Hence

$$-(x - y)(1 - F(y)) \leq T_1(x) - T_1(y) \leq -(x - y)(1 - F(x)), \tag{A.1}$$

so

$$(x - y)F(y) \leq T_1(x) + x - T_1(y) - y \leq (x - y)F(x). \tag{A.2}$$

(a) Immediate from the fact that $\max\{\xi - x, 0\}$ is continuous, nonincreasing, and convex in x for any ξ.

(b) Let $x \geq b$. Then $\max\{\xi - x, 0\} = 0$ for $\xi \leq b$, so $\xi \leq b \leq x$. Since $f(\xi) = 0$ for $\xi > b$ from Eq. (2.2), we get $T_1(x) = \int_0^b \max\{\xi - x, 0\}f(\xi)d\xi = \int_0^b 0f(\xi)d\xi = 0$; the later half holds. Let $y < x < b$. Then $-(x - y)(1 - F(x)) < 0$ since $F(x) < 1$ due to Eq. (2.1), so $T_1(x) < T_1(y)$ from Eq. (A.1), i.e., $T_1(x)$ is strictly decreasing on $(-\infty, b)$, hence on $(-\infty, b]$ due to (a) and Proposition A.1. Since $T_1(b) = 0$, we have $T_1(x) > T_1(b) = 0$ for $x < b$, so the former half is true.

(c) Let $y < x$. Then since $(x - y)F(y) \geq 0$, we have $T_1(y) + y \leq T_1(x) + x$ from Eq. (A.2), hence $T_1(x) + x$ is nondecreasing on $(-\infty, \infty)$. Let $a < y < x$. Then $(x - y)F(y) > 0$ due to Eq. (2.1). Thus $T_1(y) + y < T_1(x) + x$ from Eq. (A.2) or equivalently $T_1(x) + x$ is strictly increasing on (a, ∞), hence on $[a, \infty)$ due to the monotonicity of $T_1(x) + x$ and Proposition A.1. Let $x \leq a$. Then since $f(\xi) = 0$ for $\xi < x \leq a$ from Eq. (2.2) and since $\max\{\xi - x, 0\} = \xi - x$ for $a \leq \xi$, we have $T_1(x) = \int_a^\infty \max\{\xi - x, 0\}f(\xi)d\xi = \int_a^\infty (\xi - x)f(\xi)d\xi = \mu - x$. Hence since $T_1(a) = \mu - a$, if $a < x$, from the monotonicity of $T_1(x) + x$ on $[a, \infty)$ we have $T_1(x) + x > T_1(a) + a = \mu - a + a = \mu$, so $T_1(x) > \mu - x$. Thus the latter half is true. ▮

Lemma A.3

(a) *$T_c(x)$ is continuous, nonincreasing, and convex on $(-\infty, \infty)$.*

(b) *$T_c(x) > 0$ on $(-\infty, b^\circ)$ and $T_c(x) = 0$ on $[b^\circ, \infty)$ where $b^\circ \leq b$.*

(c) *$T_c(x) + x$ is nondecreasing on $(-\infty, \infty)$.*

(d) *$T_c(x) = \mu_1 - x$ on $(-\infty, a^\circ]$ and $T_c(x) > \mu_1 - x$ on (a°, ∞).*

Proof. (a) Immediate from Lemma A.2(a) and Eq. (3.3).

(b) First, note that $\mu_1 = \mu_0\mu \cdots (1^*)$ from Eq. (2.5). Since $0 < a$ by assumption, we have $T_1(0) = \mu$ due to Lemma A.2(c). From this result and the definition of $\mathbf{E}_\alpha[\alpha] = \mu_0$ we have $T_c(0) = \mathbf{E}_\alpha[\alpha T_1(0)] = \mu_0\mu = \mu_1 > 0 \cdots (2^*)$. If $x \geq b$, then $x/\alpha \geq b$ due to $\alpha \in (0, 1]$, hence $T_1(x/\alpha) = 0$ from Lemma A.2(b), so $T_c(x) = 0$ due to Eq. (3.3). From this result, (a), and the fact that $T_c(0) > 0$ due to (2^*), there exists a supremum b° of x such that $T_c(x) > 0$ (see Eq. (3.9)). Thus $T_c(x) > 0$ for $x < b^\circ$ and $T_c(x) = 0$ for $x \geq b^\circ$. Since $T_c(b) = \mathbf{E}_\alpha[\alpha T_1(b/\alpha)] = 0$ due to $b/\alpha \geq b$ and Lemma A.2(b), we have $b^\circ \leq b$.

(c) Let $y < x$. Then noting Eqs. (3.3) and (A.2), we get

$$\begin{aligned} T_{\mathrm{c}}(x) + x - T_{\mathrm{c}}(y) - y &= \mathbf{E}_\alpha[\alpha\big(T_1(x/\alpha) + x/\alpha - T_1(y/\alpha) - y/\alpha\big)] \\ &\geq \mathbf{E}_\alpha[\alpha(x/\alpha - y/\alpha)F(y/\alpha)] = (x-y)\,\mathbf{E}_\alpha[F(y/\alpha)] \geq 0, \end{aligned}$$

so that $T_{\mathrm{c}}(y) + y \leq T_{\mathrm{c}}(x) + x$. Thus $T_{\mathrm{c}}(x) + x$ is nondecreasing on $(-\infty, \infty)$.

(d) Define $G(x) = T_{\mathrm{c}}(x) + x - \mu_1$. Let $x \leq 0$. Then since $0 < a$ by assumption, we get $x/\alpha \leq 0 < a$ for $\alpha \in (0,1]$, so $T_1(x/\alpha) = \mu - x/\alpha$ from Lemma A.2(c). Hence, from Eq. (3.3) and (1*) we get $T_{\mathrm{c}}(x) = \mathbf{E}_\alpha[\alpha(\mu - x/\alpha)] = \mathbf{E}_\alpha[\alpha\mu - x] = \mu_0\mu - x = \mu_1 - x$ or equivalently $G(x) = 0 \cdots (3^*)$ for $x \leq 0$. Since $\mu > a$ by assumption, we have $\mu/\alpha > a$ for $\alpha \in (0,1]$, hence $T_1(\mu/\alpha) > \mu - \mu/\alpha$ from Lemma A.2(c), so $\alpha T_1(\mu/\alpha) > \alpha\mu - \mu$ for $\alpha \in (0,1]$. Thus $T_{\mathrm{c}}(\mu) = \mathbf{E}_\alpha[\alpha T_1(\mu/\alpha)] > \mathbf{E}_\alpha[\alpha\mu - \mu] = \mu_0\mu - \mu = \mu_1 - \mu$ or equivalently $G(\mu) > 0 \cdots (4^*)$. In addition, since $G(x)$ is nondecreasing on $(-\infty,\infty)$ from (c), noting (3*) and (4*), we see that there exists a maximum a° of x such that $G(x) = 0$, i.e., $T_{\mathrm{c}}(x) = \mu_1 - x$ (see Eq. (3.8)). Thus, $G(x) > 0$ for $x > a^\circ$, so $T_{\mathrm{c}}(x) > \mu_1 - x$ for $x > a^\circ$; and $G(x) = 0$ for $x \leq a^\circ$, so $T_{\mathrm{c}}(x) = \mu_1 - x$ for $x \leq a^\circ$. ▮

Proof of Lemma 3.2

(a) Immediate from Eq. (3.7), Lemmas A.1(a), and A.3(a).

(b) Note that $b^\circ \leq b$ from Lemma A.3(b). Suppose $b^\circ = b$. Then the assertion clearly holds due to Lemmas A.1(b) and A.3(b). Suppose $b^\circ < b$. Let $x < b^\circ$. Then since $T_{\mathrm{p}}(x) > 0$ due to Lemma A.1(b) and $x < b^\circ < b$, and since $T_{\mathrm{c}}(x) > 0$ due to Lemma A.3(b), we have $\mathcal{T}(x) > 0$. Let $b^\circ \leq x < b$. Then since $T_{\mathrm{p}}(x) > 0$ due to Lemma A.1(b) and since $T_{\mathrm{c}}(x) = 0$ due to Lemma A.3(b), it follows that $\mathcal{T}(x) = \max\{T_{\mathrm{c}}(x), T_{\mathrm{p}}(x)\} = T_{\mathrm{p}}(x) > 0$. Thus the former half of the assertion holds. Let $x \geq b$. Since $x \geq b > b^\circ$, we have $T_{\mathrm{p}}(x) = 0$ due to Lemma A.1(b) and since $T_{\mathrm{c}}(x) = 0$ due to Lemma A.3(b), so $\mathcal{T}(x) = 0$, the latter half holds.

(c) Since $\lambda T_{\mathrm{p}}(x) + x = \lambda(T_{\mathrm{p}}(x) + x) + (1-\lambda)x$ and $\lambda T_{\mathrm{c}}(x) + x = \lambda(T_{\mathrm{c}}(x) + x) + (1-\lambda)x$, it follows from Lemmas A.1(c), A.3(c), and the assumption of $\lambda < 1$ that $\lambda T_{\mathrm{p}}(x) + x$ and $\lambda T_{\mathrm{c}}(x) + x$ are both strictly increasing on $(-\infty, \infty)$. Therefore, the assertion is evident from the fact that $\lambda\mathcal{T}(x) + x = \max\{\lambda T_{\mathrm{c}}(x) + x, \lambda T_{\mathrm{p}}(x) + x\}$.

(d) Let $x \leq (\geq)\, y$. Then $\mathcal{T}(x) - \mathcal{T}(y) \geq (\leq)\, 0$ from (a). Since $1 - F(x) \leq 1$ and $1 - F(y) \leq 1$, from Eq. (A.1) we get $T_1(x) - T_1(y) \leq (\geq)\, y - x$ if $x \leq (\geq)\, y$. In addition, since $x/\alpha \leq (\geq)\, y/\alpha$ due $\alpha \in (0,1]$, from Eq. (3.3) we get

$$\begin{aligned} T_{\mathrm{c}}(x) - T_{\mathrm{c}}(y) &= \mathbf{E}_\alpha[\alpha(T_1(x/\alpha) - T_1(y/\alpha))] \\ &\leq (\geq)\ \mathbf{E}_\alpha[\alpha(y/\alpha - x/\alpha)] = \mathbf{E}_\alpha[y - x] = y - x. \end{aligned}$$

Further, noting Eq. (3.7) and Lemma A.1(d), if $x \leq y$, we obtain

$$\begin{aligned} \mathcal{T}(x) - \mathcal{T}(y) &= \max\{T_{\mathrm{c}}(x), T_{\mathrm{p}}(x)\} - \max\{T_{\mathrm{c}}(y), T_{\mathrm{p}}(y)\} \\ &\leq \max\{T_{\mathrm{c}}(x) - T_{\mathrm{c}}(y), T_{\mathrm{p}}(x) - T_{\mathrm{p}}(y)\} \leq \max\{y - x, y - x\} = y - x. \end{aligned}$$

Multiplying both sides of the above inequality by -1 leads to $\mathcal{T}(y)-\mathcal{T}(x) \geq x-y$ for $x \leq y$, and then interchanging the notations x and y yields $\mathcal{T}(x) - \mathcal{T}(y) \geq y - x$ for $y \leq x$. ▮

B. Lemma 3.3

(a) The assertion is proven by the subsequent three steps below.

S1. First, let us prove that $x^\star$ is finite and that $x^\star < a$.

1. Let $h^\star = \sup_{z>a} h(z)$ where $h(z) = p(z)(z-a)/(1-p(z))$ for $z > a$ (, hence $p(z) < 1$ due to Eq. (2.3 (2))). Note that $\underline{f} > 0$ from Eq. (2.4). Owing to Eq. (2.3 (2, 3)) we have $0 < p(b-\varepsilon) < 1$ for an infinitesimal $\varepsilon > 0$ such that $b > b-\varepsilon > a$. Hence $h(b-\varepsilon) = p(b-\varepsilon)(b-\varepsilon-a)/(1-p(b-\varepsilon)) > 0$, implying that $h^\star \geq h(b-\varepsilon) > 0 \cdots (1^*)$. Assume that $h^\star = \infty$. Now $h(z) = 0$ for $z \geq b$ since $p(z) = 0$ due to Eq. (2.3 (4)), hence $h^\star = \sup_{b>z>a} h(z)$. Accordingly, the assumption of $h^\star = \infty$ implies that there exists at least one z' such that $a < z' < b$ and $h(z') \geq H/\underline{f}$ for any given sufficiently large $H > 1$, i.e., $h(z') = p(z')(z'-a)/(1-p(z')) \geq H/\underline{f}$. Then
$$p(z')(z'-a) = (1-p(z'))h(z') \geq (1-p(z'))H/\underline{f} = \Pr\{\xi < z'\}H/\underline{f} \cdots (2^*).$$
Since $f(\xi) \geq \underline{f}$ for $a \leq \xi \leq z' < b$ due to Eq. (2.4), we have $\Pr\{\xi < z'\} = \int_a^{z'} f(\xi)d\xi \geq \underline{f}\int_a^{z'} d\xi = (z'-a)\underline{f}$. Hence, from (2^*) we have $p(z')(z'-a) \geq (z'-a)\underline{f}H/\underline{f} = (z'-a)H$, leading to the contradiction of $p(z') \geq H > 1$. Accordingly it must follows that $h^\star < \infty \cdots (3^*)$.
2. Using the above result, let us prove that $x^\star$ is finite. For convenience, let us further define $T_{\mathrm{p}}(x,z) = p(z)(z-x)$ for any x, hence $T_{\mathrm{p}}(x) = \max_z T_{\mathrm{p}}(x,z)$. Here note that $T_{\mathrm{p}}(x) = \max_{z\geq a} T_{\mathrm{p}}(x,z) \cdots (4^*)$ due to Lemma 3.1(a). Then for any given x let us consider the four successive assertions:
$$A_1\langle z(x) > a\rangle,$$
$$A_2\langle T_{\mathrm{p}}(x,a) < T_{\mathrm{p}}(x,z') \text{for at least one } z' > a\rangle,$$
$$A_3\langle a - h(z') < x \text{for at least one } z' > a\rangle,$$
$$A_4\langle \inf_{z>a}\{a-h(z)\} < x\rangle.$$
Here note that $1 > p(z)$ for any $z > a$ from Eq. (2.3 (2)) and that $z(x) \geq a$ for all x due to Lemma 3.1(a).
   a. Suppose A_1 is true.
      i. Let $x \geq b$, so $x \geq b > a$, hence $a-x < 0$. Then $z(x) = b > a$ from Lemma 3.1(b). Here note that $p(a) = 1$ from Eq. (2.3 (1)) and $p(z(x)) = 0$ from Eq. (2.3 (4)). Since $T_{\mathrm{p}}(x,a) = p(a)(a-x) = a-x < 0$ and $T_{\mathrm{p}}(x,z(x)) = p(z(x))(z(x)-x) = 0$, we have $T_{\mathrm{p}}(x,a) < T_{\mathrm{p}}(x,z(x))$, implying that A_2 holds, i.e., $A_1 \Rightarrow A_2$.
      ii. Let $x < b$. Assume that $T_{\mathrm{p}}(x,a) \geq T_{\mathrm{p}}(x,z')$ for all $z' > a$, so for $z' \geq a$, implying that $z(x) = a$ from (4^*), which contradicts A_1. Hence it must be that $T_{\mathrm{p}}(x,a) < T_{\mathrm{p}}(x,z')$ for at least one $z' > a$, thus A_2 must be true, i.e., $A_1 \Rightarrow A_2$.

Accordingly, it follows that $A_1 \Rightarrow A_2$ whether $x \geq b$ or $x < b$. Suppose A_2 is true. Then if $z(x) = a$, we have $T_{\mathrm{p}}(x,a) < T_{\mathrm{p}}(x,z') \leq T_{\mathrm{p}}(x) = T_{\mathrm{p}}(x,z(x)) = T_{\mathrm{p}}(x,a)$, which is a contradiction, hence it must be that $z(x) > a$ due to Lemma 3.1(a), i.e., A_1 is true, so that $A_2 \Rightarrow A_1$. From all the above it eventually follows that $A_1 \Leftrightarrow A_2$.

b. Since $p(a) = 1$ from Eq. (2.3 (1)), for any given $z' > a$ (, hence $1 > p(z')$ from Eq. (2.3 (2))) we have

$$\begin{aligned}
T_{\mathrm{p}}(x,a) - T_{\mathrm{p}}(x,z') &= p(a)(a-x) - p(z')(z'-x) \\
&= a - x - p(z')((a-x) + (z'-a)) \\
&= (1-p(z'))(a-x) - p(z')(z'-a) \\
&= (1-p(z'))\big(a - x - p(z')(z'-a)/(1-p(z'))\big) \\
&= (1-p(z'))(a - h(z') - x),
\end{aligned}$$

hence it can be immediately seen that $A_2 \Leftrightarrow A_3$.

c. If A_3 is true, then clearly so also is A_4, i.e., $A_3 \Rightarrow A_4$. If A_4 is true, then $a - h(z') < x$ for at least one $z' > a$, hence A_3 is true, i.e., $A_4 \Rightarrow A_3$, so that $A_3 \Leftrightarrow A_4$.

Since $A_1 \Leftrightarrow A_4$ from all the above, we eventually obtain

$$\begin{aligned}
x^\star &= \inf\{x \mid z(x) > a\} \\
&= \inf\{x \mid \inf_{z>a}\{a - h(z)\} < x\} = \inf_{z>a}\{a - h(z)\} = a - \sup_{z>a} h(z) = a - h^\star \cdots (5^*),
\end{aligned}$$

which is finite due to $0 < h^\star < \infty$ from $(1^*, 3^*)$. Hence $x^\star < a$ due to (5^*).

S2. Let $x < x^\star$. Then $z(x) = a$ from the definition of $x^\star$ and Lemma 3.1(a), so $p(z(x)) = 1$ from Eq. (2.3 (1)), thus $T_{\mathrm{p}}(x) = p(z(x))(z(x) - x) = a - x$ for $x < x^\star$ (see Figure B.6). Now, $T_{\mathrm{p}}(a) > 0 = a - a$ from Lemma A.1(b); in other words, $T_{\mathrm{p}}(x) > a - x$ for $x = a$ or equivalently $T_{\mathrm{p}}(x) + x > a$ for $x = a$, implying that $a^\star < a$. Therefore, $T_{\mathrm{p}}(x) + x > a$ for any $x \geq a$ from Lemma A.1(c) or equivalently $T_{\mathrm{p}}(x) > a - x$ for $x \geq a$. Further, for any x we have $T_{\mathrm{p}}(x) \geq p(a)(a-x) = a - x$ due to $p(a) = 1$ from Eq. (2.3 (1)). Consequently, it follows that there exists an infimum of x such that $T_{\mathrm{p}}(x) > a - x$, i.e., $a^\star$ defined by Eq. (3.8). Hence $x^\star \leq a^\star$ because if $a^\star < x^\star$, then for $a^\star < x < x^\star$ we obtain the contradiction of $T_{\mathrm{p}}(x) = a - x$ due to $x < x^\star$ and $T_{\mathrm{p}}(x) > a - x$ due to $a^\star < x$. Thus $x^\star \leq a^\star < a$. Accordingly, the former half of the assertion holds. From the above and the definition of $a^\star$ given by Eq. (3.8), it follows that $T_{\mathrm{p}}(x) = a - x$ on $(-\infty, a^\star]$ and $T_{\mathrm{p}}(x) > a - x$ on $(a^\star, \infty) \cdots (6^*)$.

S3. Now since $T_{\mathrm{c}}(a^\circ) = \mu_1 - a^\circ$ from Lemma A.3(d), if $\mu_1 < a^\circ$, then $T_{\mathrm{c}}(a^\circ) < 0$, which contradicts Lemma A.3(b). Hence, it must be that $\mu_1 \geq a^\circ$. In addition, since $0 = T_{\mathrm{c}}(b^\circ) \geq \mu_1 - b^\circ$ from Lemma A.3(b,d), we get $b^\circ \geq \mu_1$, hence $a^\circ \leq b^\circ$. Therefore, noting $c^\circ = \min\{x^\star, a^\circ\}$, if $a^\circ \geq x^\star$, then $c^\circ = x^\star \leq a^\circ \leq b^\circ$, and if $a^\circ < x^\star$, then $c^\circ = a^\circ \leq b^\circ$. Thus $c^\circ \leq b^\circ$ whether $a^\circ \geq x^\star$ or $a^\circ < x^\star$. Since $b^\circ \leq b$ from Lemma A.3(b), we have $c^\circ \leq b^\circ \leq b$. Accordingly, the later half holds.

(b) Let $x \leq c^\circ$, so $x \leq x^\star$ and $x \leq a^\circ$ due to Eq. (3.9). Then $T_{\mathrm{p}}(x) = a - x$ from $x \leq x^\star \leq a^\star$ due to (a) and from (6^*). In addition, owing to $x \leq a^\circ$, we get $T_{\mathrm{c}}(x) = \mu_1 - x$ from Lemma A.3(d).

Therefore, from Eq. (3.6) we obtain $J(x) = \mu_1 - x - (a - x) = \mu_1 - a$ for $x \le c^\circ$. Thus the former half holds. Let $x \ge b$. Then $T_c(x) = 0$ from Lemma A.3(b) and $T_p(x) = 0$ from Lemma A.1(b), so $J(x) = 0$. Hence the latter half is true.

(c) Let $b^\circ < b$. If $b^\circ \le x < b$, then $T_c(x) = 0$ from Lemma A.3(b) and $T_p(x) > 0$ from Lemma A.1(b). Hence we get $J(x) = -T_p(x) < 0$ from Eq. (3.6). Since $T_p(x)$ is strictly decreasing on $[b^\circ, b)$ due to Lemma A.1(a), it follows that $J(x)$ is strictly increasing on $[b^\circ, b)$. ▮

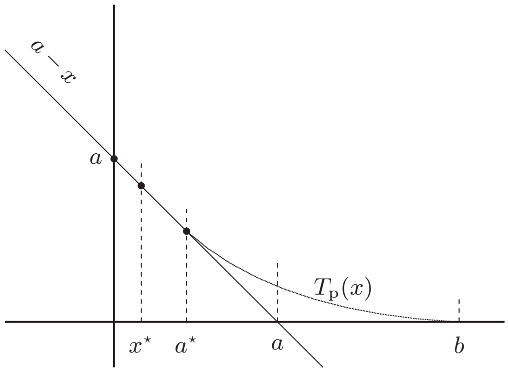


Figure B.6: $x^\star \le a^\star < a$.

C. Lemma 6.1

(a) Since $U_0(0) = M \ge \rho = U_0(i) \cdots (1^*)$ for $i \ge 1$ by assumption, Eqs. (4.8), and (4.5), the former half of the assertion is clearly true for $t = 0$. Now $U_0(1) \le M = U_0(0)$ from (1*). Let $U_{t-1}(1) \le M = U_{t-1}(0)$. Since $\mathcal{T}(U_{t-1}(0)) \ge 0$ for $t \ge 1$ due to Lemma 3.2(b), we have $U_t(1) \le \lambda\mathcal{T}(U_{t-1}(1)) + U_{t-1}(1) \le \lambda\mathcal{T}(M) + M = M = U_t(0)$ for $t \ge 1$ from Eq. (4.10), Lemma 3.2(c,b), and Eq. (4.8). Hence by induction it follows that $U_t(1) \le U_t(0) \cdots (2^*)$ for $t \ge 0$. Suppose $U_{t-1}(i)$ are nonincreasing in $i \ge 0$. Since $U_{t-1}(i) \le U_{t-1}(i-1) \le U_{t-1}(i-2)$ for $i \ge 2$, from Eq. (4.10) we get

$$\begin{aligned} U_t(i) - U_t(i-1) \;=\; & \lambda\big(\mathcal{T}(U_{t-1}(i)) - \mathcal{T}(U_{t-1}(i-1)) + \mathcal{T}(U_{t-1}(i-2)) - \mathcal{T}(U_{t-1}(i-1))\big) \\ & + U_{t-1}(i) - U_{t-1}(i-1) \cdots (3^*), \qquad t \ge 1, \quad i \ge 2. \end{aligned}$$

Note that $\mathcal{T}(U_{t-1}(i)) - \mathcal{T}(U_{t-1}(i-1)) \le U_{t-1}(i-1) - U_{t-1}(i)$ due to Lemma 3.2(d) and $\mathcal{T}(U_{t-1}(i-2)) - \mathcal{T}(U_{t-1}(i-1)) \le 0$ due to Lemma 3.2(a). Accordingly, from (3*) we get

$$\begin{aligned} U_t(i) - U_t(i-1) \;\le\; & \lambda(U_{t-1}(i-1) - U_{t-1}(i)) + U_{t-1}(i) - U_{t-1}(i-1) \\ =\; & (1-\lambda)(U_{t-1}(i) - U_{t-1}(i-1)) \le 0, \end{aligned}$$

hence $U_t(i) \le U_t(i-1)$ for $i \ge 2$, so for $i \ge 1$ due to (2*). Thus, $U_t(i) \le U_t(i-1) \cdots (4^*)$ for $i \ge 1$ and $t \ge 0$ by induction; accordingly, the former half of the assertion holds. Now, since $U_t(0) = M$ for $t \ge 0$ from Eq. (4.8), the later half of the assertion holds for $i = 0$. Since $U_{t-1}(i) \le U_{t-1}(i-1)$ for $i \ge 1$ and $t \ge 1$ from (4*), we have $\mathcal{T}(U_{t-1}(i)) \ge \mathcal{T}(U_{t-1}(i-1))$ due to Lemma 3.2(a). Therefore, from Eq. (4.10) we get $U_t(i) \ge U_{t-1}(i)$ for $t \ge 1$ and $i \ge 1$, hence also for $t \ge 1$ and

$i \geq 0$. Consequently, the later half holds.

(b) From (1*) and Eq. (4.8) we have $U_0(i) \leq M$ for $i \geq 0$. Suppose $U_{t-1}(i) \leq M$ for $i \geq 0$. Then from Lemma 3.2(c) and the fact that $\mathcal{T}(M) = 0$ due to Lemma 3.2(b) we have $\lambda\mathcal{T}(U_{t-1}(i)) + U_{t-1}(i) \leq \lambda\mathcal{T}(M) + M = M$ for $i \geq 0$. Since $\mathcal{T}(U_{t-1}(i-1)) \geq 0$ for $i \geq 1$ due to Lemma 3.2(b), from Eq. (4.10) we have $U_t(i) \leq M$ for $i \geq 1$, hence also for $i \geq 0$ due to $U_t(0) = M$ for $t \geq 0$ from Eq. (4.8). Consequently, by induction $U_t(i)$ is upper bounded in i and t. Further, since $U_t(0) = M > \rho$ for $t \geq 0$ from Eq. (4.8) and the assumption of $\rho < M$ and since $U_t(i) \geq U_0(i) = \rho$ for $t \geq 0$ and $i \geq 1$ from (a) and Eq. (4.5), we have $U_t(i) \geq \rho$ for $t \geq 0$ and $i \geq 0$, i.e., $U_t(i)$ is lower bounded in i and t. From the above and the monotonicity of $U_t(i)$ in $t \geq 0$ for $i \geq 0$ due to (a) it follows that $U_t(i)$ converges to a finite $U(i)$ for $i \geq 0$ as $t \to \infty$. Accordingly, from Eq. (4.10) we can easily show that $U(i) = \lambda\big(\mathcal{T}(U(i)) - \mathcal{T}(U(i-1))\big) + U(i)$, hence $\mathcal{T}(U(i)) = \mathcal{T}(U(i-1))$ for $i \geq 1$. Since $\mathcal{T}(U(0)) = \mathcal{T}(M) = 0$ from Eq. (4.8) and Lemma 3.2(b), we have $\mathcal{T}(U(i)) = 0$ for $i \geq 0$. Accordingly, from Lemma 3.2(b) we obtain $U(i) \geq b$ for $i \geq 0$.

(c) Let $\rho < b$. Then $U_0(i) = \rho < b$ for $i \geq 1$ from Eq. (4.5). Suppose $U_{t-1}(i) < b$ for $i \geq 1$. From Eq. (4.10) and Lemma 3.2(b) we have $U_t(i) = \lambda\mathcal{T}(U_{t-1}(i)) + U_{t-1}(i) - \lambda\mathcal{T}(U_{t-1}(i-1)) \leq \lambda\mathcal{T}(U_{t-1}(i)) + U_{t-1}(i)$ for $i \geq 1$. Thus $U_t(i) < \lambda\mathcal{T}(b) + b = b$ for $i \geq 1$ due to Lemma 3.2(c,b). Hence the former half holds by induction, so $U(i) \leq b$ for $i \geq 1$. From this fact and (b) we get $U(i) = b$. ∎

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